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n-CONNECTEDNESS IN PURE 2-COMPLEXES

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ABSTRACT
A graph is a 1-dimensional simplicial complex. In this work we study an interpretation of "n-connectedness" for 2-dimensional simplicial complexes. We prove a 2-dimensional analogue of a theorem by Whitney for graphs:

THEOREM (A Whitney type theorem for pure 2-complexes). *Let G be a pure 2-complex with no end-triangles. Then G is n-connected if and only if the valence of e is at least n for every interior edge e of G, and there does not exist a juncture set J of less than n edges of G.*

Examples of n-connected pure 2-complexes are then given, and some consequences are proved.

1. Introduction

The notion of "n-connectedness" in graphs is a very important one which has been studied widely. Many results exist in this area, and most of these are related to the following theorem, first stated by Menger [20] for 1-dimensional continua in 1927:

For any two non-adjacent nodes of a graph, the maximum number of independent paths joining them is equal to the minimum number of nodes needed to separate them.

A result which follows from that of Menger was published in 1932 by Whitney [23] as a theorem on graphs:

A graph with at least $n+1$ nodes is n-connected if and only if the order of the minimal cut-set in G is at least n.

Since a graph may be viewed as a 1-dimensional simplicial complex, it is quite

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natural to ask which properties of graphs can be extended to 2-dimensional simplicial complexes. In this work we study an interpretation of "n-connectedness" for 2-dimensional simplicial complexes and consider the existence of Menger — or Whitney — type theorems for such complexes.

2. Preliminaries

We begin with some preliminary definitions. The definitions for graphs can be found in [4].

A pure 2-complex is a (finite for our purposes) 2-dimensional simplicial complex where every k-simplex, with $k < 2$, is contained in a 2-simplex. Abstractly, a pure 2-complex consists of a finite non-empty set V and a collection C of non-empty subsets of V such that

(i) For each $v \in V$, the set $\{v\}$ is a member of C.
(ii) Each set in C has at most three elements.
(iii) Every non-empty subset of a member of C is in C.
(iv) Every set in C of fewer than three elements is a subset of a set in C of three elements.

A 3-element set $\{u, v, w\}$ is denoted uvw and is called a *triangle* (2-*simplex*, or 2-*cell*). A 2-element set $\{u, v\}$ is denoted uv and is called an *edge* (1-*simplex*, or 1-*cell*). A 1-element set $\{u\}$ is denoted u and is called a *vertex* (0-*simplex*, or 0-*cell*). We say two triangles are *adjacent* if they share an edge. An edge $e = uv$ is said to *lie on* triangle $t = uvw$ or we say simply e is *on* triangle $t = uvw$. Two edges are *adjacent* if they lie on the same triangle.

REMARK. Many of the following definitions for pure 2-complexes which are analogues of concepts in graphs will have the same names as their graph theory counterparts. This is to facilitate the recognition of the analogy and to avoid introducing new terms. The context should make it clear whether we are referring to a pure 2-complex or to a graph.

Let G be a pure 2-complex. A 2-*subcomplex* H of G is a 2-complex whose simplexes are also simplexes of G. Let e be an edge in G. Then $G - e$ is the pure 2-subcomplex resulting from the deletion of e, all triangles having e as an edge, and any other edge which now does not lie on any triangle. Let t be a triangle in G. Then $G - t$ is the pure 2-subcomplex resulting from the deletion of t, and any edge which now does not lie on any triangle. t (or e) is said to be *removed* from G. A *path* α (or 2-*walk* in Harary and Palmer [14]) in a pure 2-complex G is an alternating sequence of edges and triangles, $e_0, t_1, e_1, t_2, e_2, \dots, t_n, e_n$, beginning and ending with edges, such that triangle t_i contains the two edges immediately preceding and following it, and all the e_i and t_i are distinct. The *length* of a path α is the number of triangles which are in α. For e and f edges of G, the *distance between* e and f, denoted $\operatorname{dist}(e,f)$, is the minimum length among paths joining e and f. A path α joining e and f will be denoted $\alpha : e - f$. If $n \geq 3$ and $e_0 = e_n$ in the above sequence, then α is a *cycle*. A pure 2-complex G is *connected* if every pair of distinct edges are joined by a path. Realizing that a path is a set of edges and triangles, we say that two paths α, β from edge e to edge f in G are *independent* if α and β only have two elements e, f in their intersection. (Note, for example, that a triangle t may be in α while an edge g of t may be in β where α and β are independent.)

For e an edge of a pure 2-complex G, the number of triangles of G which contain e as an edge is called the *triangle-valence* of e, or simply the *valence of* e, and will be denoted $\operatorname{val}(e)$. An edge e in G is a *boundary edge* of G if $\operatorname{val}(e) = 1$. If an edge e in G has triangle-valence > 1, then it is an *interior edge* of G. A triangle t in G is an *end-triangle* of G if more than one of the edges of t are boundary edges of G. We let

$$\operatorname{val}(G) = \min\{\operatorname{val}(e) \,|\, e \text{ an interior edge of } G\}.$$

An edge j of a pure 2-complex G is called a *juncture edge* if there exist two *interior edges* e, f (neither is j) in G such that every path from e to f contains j. (Clearly, no boundary edge can be a juncture edge.)

REMARK. The notion of a "cut-point p" in a connected graph G has two equivalent meanings:

(i) The removal of p and all edges incident to p causes G to be not connected.

(ii) There exist some two vertices u, v in G such that every path in G from u to v contains p.

"Juncture edge" is the analogue of meaning (ii) of "cut-point" in a graph. Note the use of "interior edges" in the definition of "juncture edge". The exclusion of boundary edges here, and of end-triangles in much of what follows, will enable us to avoid certain degenerate cases.

A pure 2-complex G with no end-triangles is *n-connected* ($n \geq 1$) if for each pair of interior edges e, f in G, there exists at least n independent paths joining e and f. A set J of triangles and/or edges of a pure 2-complex G is a *juncture set for* G if there exist in G two interior edges e, f not in J such that every path from e

to f in G contains some element of J. In this case we also say that J is a *juncture set for* e, f or that J *separates* e and f.

The *order of* J, denoted $|J|$, is the number of elements in J. We call J a *juncture set of edges* for G if J consists only of edges of G. We say J is a *minimal juncture set of edges of* G if J is a juncture set of edges for G and no set of fewer edges is a juncture set for G. Let H be a graph and let B be a set of nodes of H. We say H is *n-connected in B* if for every pair of nodes u, v of B, there exist n independent paths in H joining u and v.

We end this preliminary section with a construction which we will need in the sequel. Let G be a pure 2-complex. We construct a graph, $\bar{G}$, called the *Associated Bipartite Graph* (ABG) of G:

Let $\bar{G}$ be the bipartite graph with one class T of nodes representing the triangles of G and the other class E of nodes representing the interior edges of G. Let the node in $\bar{G}$ representing the edge (or triangle) y in G be denoted $\bar{y}$. A node $\bar{t}$ in T is joined to a node $\bar{e}$ in E if and only if e is an edge on triangle t in G. If $\alpha = \{e_0, t_1, e_1, \ldots, t_n, e_n\}$ is a path in G, then $\bar{\alpha}$ denotes the path $\{\bar{e}_0, \bar{t}_1, \bar{e}_1, \ldots, \bar{t}_n, \bar{e}_n\}$ in $\bar{G}$. (The e_i's are interior edges of course.)

The following is the version of Whitney's Theorem for graphs that will actually prove useful for us later.

THEOREM 1 (Whitney). *Let B be a subset of the nodes of a graph H, where H has at least $n+1$ nodes. Then H is n-connected in B if and only if there does not exist a cut-set of less than n nodes in H for any pair of nodes in B (i.e., there does not exist a cut-set of less than n nodes for B).*

3. A Whitney-type theorem

We begin with an easy but useful proposition whose proof is immediate.

PROPOSITION 1. *If G is a pure 2-complex with no end-triangles, then $\bar{G}$, the ABG of G, is n-connected in E if and only if G is n-connected.*

A juncture set in a pure 2-complex G with no end-triangles corresponds in $\bar{G}$ to a cut-set for E, and vice versa. Thus by considering $\bar{G}$, Proposition 1, and Theorem 1 we get the following theorem:

THEOREM 2. *Let G be a pure 2-complex with no end-triangles. Then G is n-connected if and only if there does not exist a juncture set (of triangles and/or edges) for G of order $< n$.*

As noted before, this theorem is just a rewording of some graph theory facts. It is not an exact analgoue of Whitney's Theorem, and furthermore, it is difficult to apply. A more useful result is given by the following theorem.

THEOREM 3 (A Whitney-type theorem for pure 2-complexes). *Let G be a pure 2-complex with no end-triangles. Then G is n-connected if and only if* $\mathrm{val}(e) \geq n$ *for every interior edge e of G, and there does not exist a juncture set J* of edges only *of G where* $|J| < n$.

The non-trivial direction of Theorem 3 will be proved by showing that the existence of a juncture set of order $< n$, together with the condition on edge valences, implies the existence of a juncture set *of edges* of order $< n$. This and Theorem 2 lead us to our result. First we need a lemma.

LEMMA 1. *Let G be a pure 2-complex with no end-triangles. Let e,f be two interior edges of G. Let J be a juncture set (of triangles and/or edges) for e,f* such that neither e nor f lies on any triangle of J. *Then there exists in G a juncture set* J' *of* edges only *for e and f such that* $|J'| \leq |J|$.

PROOF. Let $J = \{a_1, \ldots, a_p\}$ where $a_i = t_i$, a triangle of G, for $1 \leq i \leq k$, and $a_i = e_i$, an edge of G, for $k+1 \leq i \leq p$. (If J has no triangles, there is nothing to prove.) Without loss of generality, we can assume that no proper subset of J is a juncture set for e,f. There exists a set of p paths from e to f such that for each element a_j of J, there is one path α_j in the set which contains only a_j. For each triangle $t_j \in J$ $(1 \leq j \leq k)$, we wish to replace t_j by an edge e_j, chosen so that $\{e_1, e_2, \ldots, e_k, e_{k+1}, \ldots, e_p\}$ will be the J' desired.

For $t_j \in J$, let i_j denote the edge at which α_j enters t_j; let o_j denote the edge at which α_j exits t_j, and c_j denote the third edge of t_j. We replace t_j by an edge e_j in this way:

Let $e_j = o_j$ if o_j is the only edge of t_j
($\uparrow$) "through which there exists a path from t_j to f not containing any other element of J."

(We refer to an edge of t_j that satisfies ($\uparrow$) as an *outedge* of t_j.) Let $e_j = i_j$ if there exist two outedges (namely c_j and o_j). The edge i_j cannot be an outedge of t_j as this would give us a path in G from e to f containing no element of J. Note that $e_1, e_2, \ldots, e_k$ may not all be distinct. We wish to show that $J' = \{e_1, \ldots, e_k, e_{k+1}, \ldots, e_p\}$ is a juncture set of edges for e,f, thus proving the lemma.

Let β be a path in G from e to f. Assume β does not contain any of the edges

$e_{k+1},\ldots,e_p$. Assume β contains triangles $t_1,\ldots,t_m$ $(1 \leq m \leq k)$ of J. We claim that β contains an element of J'. Renumber the $t_1,\ldots,t_m$ in β so that they are arranged in the order of their appearance in β from e to f. So

$$\beta = \{e,\ldots,t_1,\ldots,t_2,\ldots,t_m,\ldots,f\}.$$

The path β exits t_m at either o_m or c_m but not at i_m. If the only outedge of t_m is o_m, then o_m was chosen as e_m, and $e_m \in \beta$. If not, there exist two outedges of t_m. If β entered t_m through i_m, then i_m was chosen as e_m, and $e_m \in \beta$. If β entered t_m through c_m (then $c_m \not\in J$), we go back to t_{m-1} and repeat this process. Note that at this stage we know that there exists a path, from f to the edge where β exits from t_{m-1}, which contains no element of J or J' and coincides with β from t_{m-1} to the edge at which β enters t_m. Suppose this process continues back to t_1. At this stage we have a path β', from f to the edge where β exits from t_1, which contains no element of J or J' and coincides with β from t_1 to the edge at which β enters t_2. If β exits t_1 through c_1, then there exist two outedges of t_1. Hence i_1 was chosen as e_1, and $e_1 \in \beta$. If β exits through o_1, we have two cases: If t_1 has two outedges, $i_1 = e_1$. If not, β must exit at o_1 which was chosen as e_1.

In all cases, there exists an element of J' in β, and therefore J' is a juncture set of edges for e and f.

Proof of Theorem 3. The case $n = 1$ is self-explanatory and needs no proof. Let $n \geq 2$.

It is clear that if there exist at least n independent paths joining any pair of interior edges, then there cannot be a juncture set of edges in G of order $< n$. Also it is immediate that $\mathrm{val}(e) \geq n$ for each interior edge e of G.

Let us now assume that $\mathrm{val}(e) \geq n$ for each interior edge of G, and that there does not exist in G any juncture set of edges of order $< n$. We will show that this implies there does not exist in G a juncture set (of triangles and/or edges) of order $< n$; this, together with Theorem 2, proves that G is n-connected.

Assume the contrary. Suppose J is a juncture set containing at least one triangle of G, and that J has k (k minimum) elements with $1 \leq k < n$. By definition, J is a juncture set for some two interior edges e,f of G. If neither e nor f lies on any triangle of J, then by Lemma 1, there exists J', a juncture set of *edges only* for e,f, of order $|J'| \leq |J| = k < n$ in G. This is a contradiction.

So suppose that e or f (or both) lies on a triangle in J. We would like to find an interior edge e' (also f' if necessary) in G having the property that e' and f (or f') are separated by a juncture set of triangles and/or edges of order $\leq k < n$, no triangle of which contains e' or f (f'). This would allow us to invoke Lemma 1 to obtain the desired contradiction.

Suppose e lies on a triangle t of J. Let st(e) denote the set of all triangles in G which have e as an edge. (Since val$(e) \geq n$, st(e) has at least n elements.) Let J^* denote the set of all elements of J and all edges in G lying on a triangle of J. We wish to pick e' to replace e. Note that if f is also on a triangle of J, it too will be replaced by an f' and that therefore if e and f are "too close", this may affect our choice of e'. Hence we consider first: case (i) dist$(e,f) > 2$: (A) If there exists a triangle $t' \in$ st(e) such that $t' \notin J$, and t' has an interior edge which is not in J^*, then we choose e' to be this interior edge (Fig. 1).

Otherwise, we proceed as follows: (B) Suppose there exists a triangle $s \in J$ which is adjacent to two triangles t_i, t_j of st(e) with these properties: Neither t_i nor t_j is in J; neither of the two interior edges a, b which s shares with t_j and t_i are in J and at least one of these edges, say a, on t_j lies on exactly one triangle s of J. We let e' be a and replace s in J by c, the third edge of s. See Fig. 2. (c, of course, may be in J already.) We then claim that $J_1 = J \cup \{c\} - \{s\}$ is a juncture set (of triangles and/or edges) for e' and f. (If a third interior edge c of s does not exist, replace s by b and e by a.)

We claim that such a triangle s must exist if (A) does not apply. Suppose st$(e) = \{t = t_1, t_2, \ldots, t_m\}$ with $m \geq n$. We assign to each t_j $(1 \leq j \leq m)$ of st(e) an element $F(t_j)$ in J in the following manner:

(1) $F(t_j) = t_j$ if $t_j \in J$.
(2) $ = g_j$ if $t_j \notin J$, and there exists an edge g_j of t_j such that $g_j \in J$.
(3) $ = s_j$ if $t_j \notin J$, no edge of t_j is in J, and t_j is adjacent to some triangle s_j in $J -$ st(e), where s_j is adjacent to no other triangle of st(e).

Otherwise, let

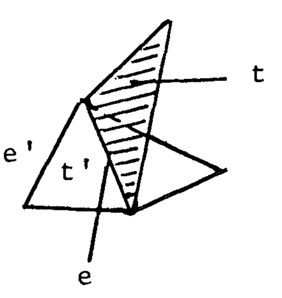

Fig. 1.

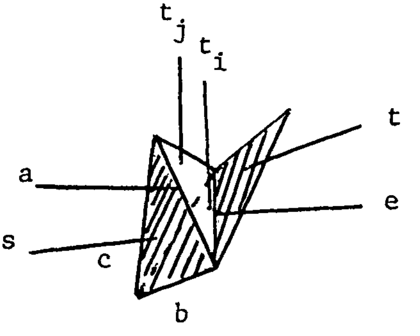

Fig. 2.

(4) $F(t_j) = s_j$ if t_j is adjacent to some triangle s_j in $J - \mathrm{st}(e)$ and s_j adjacent to a triangle $t_i \in \mathrm{st}(e)$ where $F(t_i)$ can be assigned as in (1), (2), or (3) above.

Note that all $F(t_j)$'s so far assigned are distinct elements of J, and if $F(t_j)$ can be picked this way for all j, $1 \leq j \leq m$, we have at least m elements in J, a contradiction. Hence there must exist a $t_j \in \mathrm{st}(e)$ such that $F(t_j)$ cannot be assigned according to (1), (2), (3), or (4) above. So t_j is adjacent to some triangle s in $J - \mathrm{st}(e)$ where s is adjacent to another triangle $t_i \in \mathrm{st}(e)$. First we note such an s is distinct from any element in J already assigned as in (1), (2), (3), (4). We also observe that there exists such a t_j which is adjacent to only one such s. (If not, let p be the number of t_j's for which we have not chosen $F(t_j)$ as in (1), (2), (3), (4). Then there exist at least $2p$ triangles like s, of which at least p of them are distinct. These elements of J and the $F(t_j)$'s previously assigned added up to at least n elements in J, a contradiction.) But this is precisely the situation in (B) that we desire. That is, $t_j \notin J$, t_j adjacent to a triangle $s \in J$ which is adjacent to a $t_i \in \mathrm{st}(e)$, $t_i \notin J$, and at least one of two interior edges a, b (namely a) which s shares with t_j and t_i lies on exactly one triangle s of J, with neither a nor b in J.

If e' were picked as in (A), it is clear that J is still a juncture set for e' and f. If (B) applied, then $J_1 = J \cup \{c\} - \{s\}$ is a juncture set for e' and f: Suppose there exists a path $\alpha : e' - f$ which does not contain any element of J_1. Then α does not contain c, but it must contain s — otherwise there would be a path $\alpha' : e - e' - f$ which does not contain any element of J. Hence $\alpha = \{e', \ldots, s, b, \ldots, f\}$. But then there exists a path $\alpha'' = \{e, t_i, b, \ldots, f\} : e - b - f$ which does not contain any element of J. This is a contradiction.

If necessary, f' can now be chosen in a similar manner, and we note that we may have to change our set J_1 if f' were chosen as in (B). But this new set J_2 is easily seen to be again a juncture set for e' and f'. Neither e' nor f' is in J_2^*, and $|J_2| < n$.

Case (ii) $\mathrm{dist}(e,f) < 2$, i.e. e and f lie on the same triangle t, which of course would be in J. An inspection of (A) and (B) in case (i) shows that this case can be handled no differently from case (i) because we can still choose e' first as above without any difficulties arising from the location of f. Because e' does not lie on any triangle of J (or J_1), f' can now be picked.

Case (iii) $\mathrm{dist}(e,f) = 2$: We can pick e' as in case (i) unless f is the third edge c of s (since only then is f in a crucial position — see (B) in case (i)). In that case, we choose f' first. e does not lie on any triangle of $\mathrm{st}(f)$, and e does not lie on a triangle in G which is in J and also adjacent to two triangles in $\mathrm{st}(f) - J$. Hence

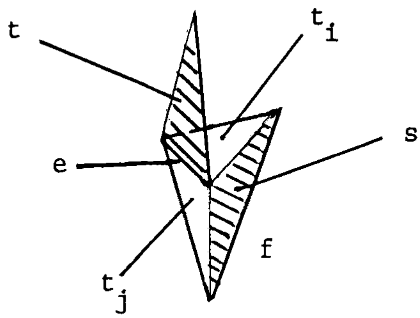

Fig. 3.

$f'(\neq e)$ can be picked as before. Since f' does not lie on any triangle of J (or the revised juncture set, if (B) applied), e' can be picked with no interference from f' (see Fig. 3).

This exhausts all the possible ways that e and f can be situated in G. Based on our earlier remarks, the proof is concluded.

4. Examples and consequences

The theorem of the previous section enables us to identify many examples of n-connected pure 2-complexes. The first of these are motivated by the work by Larman and Mani [15], in which they study a class of graphs called "n-simplicial" graphs. We parallel that effort somewhat.

Let v be a vertex of an n-dimensional simplicial complex G. The *Star Complex* v, Star(v), is the smallest subcomplex of G containing all the simplexes of G which contain v; the *Linked Complex of* v, Link(v), is the subcomplex of G consisting of all p-simplexes ($p \geq 0$) in Star(v) which do not contain v. We denote by $O_E(v)$ the set of all edges of Link(v) in a pure 2-complex. We call G 2-*radial* if G is a triangulated, compact, connected (topologically) 2-manifold without boundary. We call G n-*radial* if G is connected (topologically) and for each vertex v of G, $O_E(v) \subseteq H_v$ where H_v is an $(n-1)$-radial 2-subcomplex of G that does not contain v.

REMARK. It is clear by induction that if G is n-radial, then G has no boundary edges. In fact, every edge of G lies on at least n triangles.

The following proposition allows us to obtain many examples of n-connected pure 2-complexes.

PROPOSITION 2. *If G is n-radial, then G is n-connected.*

PROOF. It is easy to see that a triangulated, compact, connected (topologically) 2-manifold without boundary is 2-connected. Let $n \geq 3$, and assume the theorem is proven for $n-1$. Suppose G is n-radial but not n-connected. By the remark above, every edge of G has triangle-valence at least n, hence there exists a juncture set J of edges for G of order $k < n$. Let $J = \{j_1, \ldots, j_k\}$ be such a juncture set of edges for some two edges e, f of G. We can assume that no proper subset of J is a juncture set so there exist paths $\alpha_1, \alpha_2, \ldots, \alpha_k$ in G from e to f such that α_i contains only j_i from amongst $j_1, \ldots, j_k$. Consider

$$\alpha_1 = \{e = z_1, t_1, z_2, t_2, \ldots, j_1 = z_m, \ldots, z_q = f\}.$$

Let the vertices of j_1 be v_1 and v_2. There are only two ways in which α_1 behaves near j_1 (see Fig. 4).

In case (i) of Fig. 4, $z_{m-1}, z_{m+1} \in O_E(v_1) \subseteq H_{v_1}$, where H_{v_1} is an $(n-1)$-radial 2-subcomplex of G. By the inductive hypothesis, H_{v_1} is $(n-1)$-connected. Therefore, since $j_1 \not\in H_{v_i}$ and $k-1 < n-1$, there exists a path $\beta : z_{m-1} - z_{m+1}$ in H_{v_1} which avoids $j_2, \ldots, j_k$. From β, we could construct a path $\alpha : e - z_{m-1} - z_{m+1} - f$ where α contains no element of J. This is a contradiction.

In case (ii), $z_{m-1} \in O_E(v_1)$, but $z_{m+1} \not\in O_E(v_1)$. We know that z_{m+1} lies on at least $n-1$ other triangles in addition to triangle $v_1v_2v_3$. Since $|J| < n$, this implies that there exists an edge g on one of these triangles such that $g \not\in J$, and $g \in O_E(v_1)$. Hence $g \in H_{v_1}$ and there exists a path $\beta : z_{m-1} - g$ in H_{v_1} which does not contain any of $j_2, \ldots, j_k$. From β, we could construct a path $\alpha : e - z_{m-1} - g - z_{m+1} - f$ where α contains no element of J.

In either case, we see that J cannot be a juncture set for G. Therefore, by Theorem 3, G is n-connected.

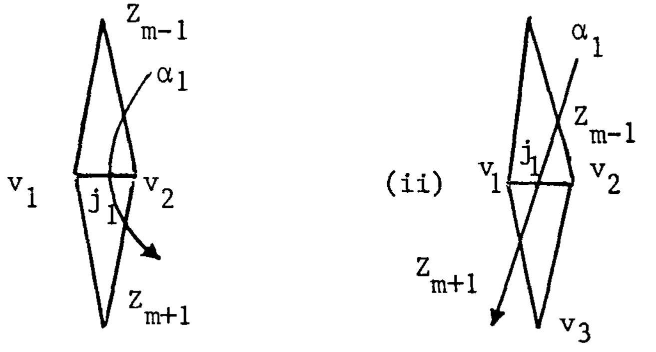

Fig. 4.

A *combinatorial* $(n-1)$-*sphere* is a simplicial complex which is piecewise affine homeomorphic to the boundary of a (geometric) n-simplex. A *combinatorial n-manifold* (without boundary) is a homogeneous (of degree n), connected (topologically) n-dimensional simplicial complex M such that for every vertex v of M, Link(v) is a combinatorial $(n-1)$-sphere.

A *polytope* is a convex hull of a finite set of points in some Euclidean space. If a polytope has dimension d, we call it a d-*polytope*. A d-polytope with $d+1$ vertices is a d-simplex. The proper faces of a d-polytope P form a cell-complex, the *boundary complex*, $B(P)$, *of* P. In particular, if P is a *simplicial polytope*, the proper faces are simplexes and the simplicial complex $B(P)$ is a combinatorial $(d-1)$-manifold.

An important result by Balinski [1] states: "If P is an n-polytope, then $\mathrm{Sk}_1(B(P))$, the graph of P, is n-connected."

In [3] Barnette extended this result to the graphs of manifolds, and in [2] he obtained further generalizations for the graphs of pseudomanifolds and of more general decompositions of homology manifolds.

Let $\mathrm{Sk}_2(M)$ denote the subcomplex of M consisting of all simplexes of M of dimension 2 or less. It is easily proved (by induction on n) that for a combinatorial n-manifold M, $\mathrm{Sk}_2(M)$ is n-radial. Thus we have the following.

THEOREM 4. *If M is a combinatorial n-manifold, then* $\mathrm{Sk}_2(M)$ *is n-connected.*

COROLLARY 1. $\mathrm{Sk}_2(A)$, *where A is an $(n+1)$-simplex, is n-connected.*

By Theorem 4, we have this corresponding result:

COROLLARY 2. *Let P be a simplicial n-polytope. Then* $\mathrm{Sk}_2(B(P))$ *is $(n-1)$-connected.*

A wide class of n-dimensional simplicial complexes which contains the class of all triangulated, compact, connected (topologically) n-manifolds (which includes the combinatorial n-manifolds) is defined as follows: A n-dimensional simplicial complex C is called a *strong n-(simplicial) complex* if C is homogeneous of degree n and for each pair f, f' of n-simplexes of C, there exists a sequence $f=f_0, g_1, f_1, g_2, \ldots, g_m, f_m=f'$ such that the g_i's are $(n-1)$-simplexes and the f_i's are n-simplexes with $f_i \cap f_{i+1} = g_{i+1}$.

We now investigate the degree of connectivity of the 2-skeleta (Sk_2) of members of this wide class of n-complexes.

PROPOSITION 3. *Let A be an $(n+1)$-simplex $(n \geq 2)$. Then the minimal juncture set of edges in* $\mathrm{Sk}_2(A)$ *has order $2n$.*

PROOF. The proof is by induction on n. When $n=2$, the 2-skeleton of the 3-simplex has 4 juncture edges in any minimal juncture set of edges. Assume the statement is true for $n-1$. Let J be a juncture set of edges for some two (non-adjacent) edges e,f of $G=\mathrm{Sk}_2(A)$, where A is an $(n+1)$-simplex. The edges e,f lie in some n-simplex $N \subseteq A$. By induction, there exist at least $2(n-1)$ edges in a minimal juncture set of edges in $\mathrm{Sk}_2(N)$. Let v be the vertex of A which is not in N. Then there exist two triangles t_1 and t_2 of $\mathrm{Sk}_2(A)$ containing v with e on t_1 and f on t_2. Let the other edges of t_1 be a and b, and let c and d be the remaining edges of t_2. The edge a is joined to one of c and d, say c, by a sequence of triangles not in $\mathrm{Sk}_2(N)$. The same is then true for b and d. It is clear then we need at least two edges to separate e,f. Hence in $\mathrm{Sk}_2(A)$, $|J| \geq 2(n-1)+2=2n$. But in fact, this order is exactly $2n$ because each edge has triangle-valence n in $\mathrm{Sk}_2(A)$.

PROPOSITION 4. *For $n \geq 3$, let C consist of two $(n+1)$-simplexes A_1, A_2 sharing an n-simplex N as a common face. Then the minimal juncture set of edges for $\mathrm{Sk}_2(C)$ has order $2n$.*

PROOF. Let J be a juncture set of j edges in $G=\mathrm{Sk}_2(C)$ which separates edges e,f. If e,f are both in the same $(n+1)$-simplex, then $j \geq 2n$ by the previous proposition. Hence we can assume that $e \in A_1$, $f \in A_2$, and neither is in N. Assume now that $j<2n$. The n-simplex N has $\binom{n+1}{2}$ edges. Since $n \geq 3$, $\binom{n+1}{2} \geq 2n$, so there exists an edge g in N which is not in J. The edges e,g are in A_1, and there exists an $e-g$ path in $\mathrm{Sk}_2(A_1)$ which does not contain any element of J ($|J|<2n$). Similarly, a $g-f$ path containing no element of J exists in $\mathrm{Sk}_2(A_2)$. Therefore, there is an $e-f$ path in $\mathrm{Sk}_2(C)$ not containing any element of J. We conclude then, that the order of a minimal juncture set of edges in $\mathrm{Sk}_2(C)$ is exactly $2n$.

THEOREM 5. *Let C be a strong $(n+1)$-simplicial complex. With $G=\mathrm{Sk}_2(C)$, let $p=\mathrm{val}(G)$ (hence $p \geq n$). Then G is k-connected where $k=\min\{p,2n\}$ ($n \geq 3$).*

PROOF. Let e,f be two edges in $\mathrm{Sk}_2(C)=G$. Then e,f lie in two $(n+1)$-simplexes A,A' respectively. If $A=A'$, by Proposition 1, the minimal juncture set in A for e,f has order $2n$. If A, A' are distinct, then there exists a sequence $S: A=A_0, N_1, A_1, \ldots, N_k, A_k=A'$ of $(n+1)$- and n-simplexes such that $A_i \cap A_{i+1}=N_{i+1}$. This sequence S can be viewed as a subcomplex of C. By Proposition 4, $\mathrm{Sk}_2(S)$ has at least $2n$ edges in each juncture set of edges.

Therefore, any juncture set in G must have at least $2n$ elements. And if $\mathrm{val}(G) = p$, with $n \le p \le 2n$, by Theorem 3 $G = \mathrm{Sk}_2(C)$ is p-connected.

REMARKS. (i) The following shows why Proposition 4 does not hold for $n = 2$: Two 3-simplexes intersecting on a 2-simplex have a juncture set of 3 edges, not $4 = 2 \cdot 2$ edges.

(ii) Not all n-connected 2-complexes are n-radial: the 2-connected 2-complex obtained by identifying a pair of vertices in a triangulated 2-sphere is not 2-radial. Note also that any n-connected 2-complex with boundary edges is not the 2-skeleton of a strong n-complex.

We conclude this section with generalizations of a few of the graph-theoretic theorems which follow from Whitney's Theorem. The generalizations of the 2-complex case can easily be established because of the existence of our Whitney-type theorem, Theorem 3 of Section 3. To facilitate reading, we put the letter "A" after the number of a theorem if it is a known theorem for graphs, and a "B" for its corresponding generalization.

THEOREM 6A (Dirac [6]). *Let G be an n-connected graph, $V = \{A_1, \dots, A_p\}$, $W = \{B_1, \dots, B_q\}$, be disjoint subsets of $N(G)$, and let $a_1, \dots, a_p$ and $b_1, \dots, b_q$ be positive integers such that $\Sigma_{i=1}^p a_i = \Sigma_{j=1}^q b_j = n$. Then there exist n independent paths from V to W such that a_i of them have A_i as an end-node and b_j of them have B_j as an end-node, for all i, j.*

THEOREM 6B. *Let G be an n-connected 2-complex, $V = \{e_1, \dots, e_p\}$, $W = \{f_1, \dots, f_q\}$ be disjoint subsets of interior edges, and let $a_1, \dots, a_p$ and $b_1, \dots, b_q$ be positive integers such that $\Sigma_{i=1}^p a_i = \Sigma_{j=1}^q b_j = n$. Then there exist in G n independent paths between V and W, a_i of which have e_i as an end-edge, and b_j of which have f_j as an end-edge, for all i, j.*

PROOF OF 6B. We make the observation that the conclusion of 6A could be established with the weakened hypothesis that "no set of fewer than n nodes separate any node of V from any node of W". Since our 2-complex is n-connected, its ABG, $\bar{G}$, is n-connected *in E*. Now apply 6A to $\bar{G}$, and 6B follows.

THEOREM 7A (Sabidussi). *If a graph G has at least $2n$ nodes, then G is n-connected if and only if for each pair of disjoint sets V, W of nodes, with $|V| = |W| = n$, there exist n disjoint paths from V to W.*

THEOREM 7B. *A pure 2-complex G with at least $2n$ interior edges and no* *end-triangles is n-connected if and only if* $\mathrm{val}(G) \geq n$, *and for any two disjoint sets V, W of n interior edges each, there exist n disjoint $V-W$ paths.*

PROOF OF 7B. If G is n-connected, existence of the paths follows from Theorem 1B. Now suppose we have a juncture set J of edges for some two interior edges e,f of G. Let $V' = \{$interior edges g in $G \mid$ there exists an $e-g$ path not containing any element of $J\}$. Let W' be the set of all interior edges of G not in J or in V'. We can assume $|J| \geq n-1$. Then there exists a partition of J into J_1 and J_2 such that $|V' \cup J_1| \geq n$ and $|W' \cup J_2| \geq n$. Since any n-element subsets $V \subseteq V' \subseteq J_1$, $W \subseteq W' \cup J_2$ are joined by n disjoint paths, each containing at least one edge of J, this implies $|J| \geq n$. Since we also have $\mathrm{val}(G) \geq n$, by Theorem 3, Section 3, G is n-connected.

THEOREM 8A (Lick [16], Mesner and Watkins [21]). *Let G be a graph with at least $n+1$ nodes ($n \geq 2$). Then G is n-connected if and only if for each set S of n nodes of G, any two nodes of S are contained in a cycle of G which avoids the other $n-2$ nodes of S.*

THEOREM 8B. *Let G be a pure 2-complex with no end-triangles. Then G is n-connected if and only if* $\mathrm{val}(G) \geq n$, *and for each set S of n interior edges, any two edges of S are contained in a cycle of G which avoids the other $n-2$ edges of S.*

PROOF OF 8B. Let G be n-connected. Let S be a set of n interior edges and let e,f be members of S. There exist at least n independent paths from e to f. Hence e,f lie on a cycle not containing any other elements of S.

To prove that G is n-connected, we prove that no set of fewer than n edges can be a juncture set for G. If such a set J exists, we can assume $|J| \geq n-1$, and J is a juncture set for some two interior edges e,f of G. Then the set $S = \{e,f\} \cup J'$, where $|J'| = n-2$, $J' \subseteq J$, has n elements. By the hypothesis, e,f lie on a cycle not containing any element of J'. Therefore $|J| \geq n$.

The following two theorems by Halin also have 2-complex analogues which can be proven without difficulty.

THEOREM 9A (Halin). *Let G be a graph with at least $n+1$ nodes ($n \geq 2$). Then G is n-connected if and only if for every n nodes $a_1, \ldots, a_n$ of G and for every ν with $2 \leq \nu \leq n$, there exists a cycle γ which contains $a_1, \ldots, a_\nu$ but avoids $a_{\nu+1}, \ldots, a_n$.*

THEOREM 9B. *Let G be a pure 2-complex with no end-triangles. Then G is n-connected if and only if* $\mathrm{val}(G) \geq n$ *and*

(*) *for every n interior edges* $e_1, \ldots, e_n$ *of G and for every* ν *with* $2 \leq \nu \leq n$, *there exists a cycle* γ *which contains* $e_1, \ldots, e_\nu$ *but does not contain* $e_{\nu+1}, \ldots, e_n$.

THEOREM 10A (Halin [10]). *Let G be a graph with at least* $n+1$ *nodes. Then G is n-connected if and only for every* $n+1$ *nodes* $e_1, \ldots, e_{n+1}$ *and* ν, $2 \leq \nu \leq n+1$, *there exists a path in G which contains* $e_1, \ldots, e_\nu$ *but avoids* $e_{\nu+1}, \ldots, e_{n+1}$.

THEOREM 10B. *Let G be a pure 2-complex with no end-triangles. Then G is n-connected if and only if* $\mathrm{val}(G) \geq n$ *and*

(*) *for every* $n+1$ *interior edges* $e_1, \ldots, e_{n+1}$ *and every* ν, $2 \leq \nu \leq n+1$, *there exists a path in G which contains* $e_1, \ldots, e_\nu$ *but not* $e_{\nu+1}, \ldots, e_{n+1}$.

REMARK. We cannot dispense with the statement "$\mathrm{val}(G) \geq n$" in either Theorem 9B or 10B. Consider the 2-complex G obtained from the complete 2-complex on vertices $\{1, 2, 3, 4, 5\}$ with the triangles 125 and 135 deleted. G is 2-connected, and the reader can check that G satisfies (*) for both Theorems 9B and 10B with $n = 3$. (For example, the cycle $\{14, 124, 24, 234, 34, 134, 14\}$ contains edges 14 and 24 but not the edge 12.) However, $\mathrm{val}(G) \geq 2$ only.

5. Further problems

Other than the ones we have already mentioned, there are many directions in which to go off at this time. We will just mention some of the more natural and more important graph theory concepts we have yet to study on a pure 2-complex.

(1) Halin [10], Watkins and Mesner [21], [22], Mader [17], [18], etc., obtained a whole battery of theorems concerning n-connected graphs. We have only barely touched on these.

(2) The relationship between the degree of connectivity in a pure 2-complex and its embeddability in a Euclidean space: A planar graph, for example, cannot have arbitrarily large connectivity. Some relationship like this almost certainly exists for pure 2-complexes.

(3) Erdös and König [14] extended Menger's Theorem to infinite graphs. Can one extend our Whitney-type theorem to infinite 2-complexes?

(4) And finally, even though our Whitney-type theorem is not immediately extendable to higher dimensional complexes, will it have an analogue in those cases if some suitable hypotheses were added?

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